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Determinism in Free Bosons

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It is shown how to map the quantum states of a system of free Bose particles one-to-one onto the states of a completely deterministic model. It is a classical field theory with a large (global) gauge group.

KEY WORDS: quantum mechanics; determinism; Bell inequalities; guage invariance.

1. INTRODUCTION

Consider a model for free, relativistic bosons, described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2. \tag{1}$$

We take this Lagrangian as our prototype, but Lorentz invariance is not crucial; with slight modifications, our analysis will be applicable just as well to nonrelativistic free bosons. If we compare it with a *classical* field theory, described by the Klein–Gordon equation

$$(\Delta - \mu^2)\varphi - \ddot{\varphi} = 0, \tag{2}$$

where the dots refer to time differentiation, then we obviously have a quite different physical system. One important difference is that phase space for the classical system is, in a sense, twice as big: at a given time $t = 0$, one may specify the values of $\varphi(\vec{x}, 0)$ and $\dot{\varphi}(\vec{x}, 0)$ independently, whereas the quantized theory only requires us to specify the operators $\phi(\vec{x}, 0)$.

The classical model is obviously invariant under the group of the following transformations:

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$$\begin{aligned}\varphi'(\vec{x},t) &= \int \mathrm{d}\vec{y}(K_1(\vec{y})\varphi(\vec{x}+\vec{y},t)+K_2(\vec{y})\dot{\varphi}(\vec{x}+\vec{y},t)),\\ \dot{\varphi}'(\vec{x},t) &= \int \mathrm{d}\vec{y}(K_1(\vec{y})\dot{\varphi}(\vec{x}+\vec{y},t)+K_2(\vec{y})(\Delta-\mu^2)\varphi(\vec{x}+\vec{y},t)),\end{aligned} \tag{3}$$

where the integration kernels $K_1(\vec{y})$ and $K_2(\vec{y})$ are arbitrary but fixed real generalized functions of $\vec{y}$, independent of $\vec{x}$, and independent of t. They are distributions, obeying certain integrability conditions. If otherwise K_1 and K_2 were essentially arbitrary, then *all* field configurations obeying the Klein–Gordon equation could be transformed into any other solution. However, we impose the following restrictions:

$$K_1(\vec{y}) = K_1(-\vec{y}); \qquad K_2(\vec{y}) = -K_2(-\vec{y}), \tag{4}$$

In momentum space, the transformation then reads

$$\begin{aligned}\hat{\varphi}'(\vec{k},t) &= \hat{K}_1(\vec{k})\hat{\varphi}(\vec{k},t)+i\hat{K}_2(\vec{k})\dot{\hat{\varphi}}(\vec{k},t),\\ \dot{\hat{\varphi}}'(\vec{k},t) &= \vec{K}_1(\vec{k})\dot{\hat{\varphi}}(\vec{k},t)-i\hat{K}_2(\vec{k})(\vec{k}^2+\mu^2)\hat{\varphi}(\vec{k},t),\end{aligned} \tag{5}$$

where $\hat{K}_1(\vec{k})$ is an even real function of $\vec{k}$, and $\hat{K}_2(\vec{k})$ an odd real function.

We now decide to call all functionals of $\varphi(\vec{x},t)$ that are invariant under the transformation (3) with the constraint (4) (which can easily be seen to form a group) "observables," whereas all noninvariant quantities are fundamentally unobservable. Notice that, because of the constraint (4), the space of observables at $t=0$ is essentially half as big as the classical phase space, just like the quantum theory. This is easily seen in the following way.

In the original system, we were free to choose $\varphi(\vec{x},0)$ and its time derivative, $\dot{\varphi}(\vec{x},0)$, independently, after which all field values at different times are fixed by the Klein–Gordon equation. But now we may impose a "gauge condition," such as

$$\dot{\varphi}(\vec{x},0) = \delta(\vec{x}) \quad \text{or} \quad \dot{\hat{\varphi}}(\vec{k}) = (2\pi)^{-3/2}. \tag{6}$$

Inspecting the second line of Eq. (5), we see that this condition for $\dot{\hat{\varphi}}'(\vec{k},0)$ can be realized starting from any set of values of $\dot{\hat{\varphi}}(\vec{k},0)$ and $\hat{\varphi}(\vec{k},0)$ by adjusting the real numbers $\hat{K}_1(\vec{k})$ and $\hat{K}_2(\vec{k})$ in there, except for the singular cases when the phases of $\dot{\hat{\varphi}}(\vec{k},0)$ and $\hat{\varphi}(\vec{k},0)$ coincide, which happens only for a set of $\vec{k}$ values of measure zero.

Hence this condition fixes[3] the kernels K_1 and K_2, so that the values of $\varphi(\vec{x},0)$ in this gauge all correspond to observables.

Not only does the counting argument for the dimensionality of phase space for this model match with the quantum mechanical case, we claim an important theorem:

[3] Observe that this gauge choice is not completely free from singularities and ambiguities, but those are not relevant for the present discussion.

The classical model where the observables are restricted to be invariant under (3), with constraints (4), is equivalent to the quantized model.

We prove this theorem by first considering a single harmonic oscillator.

If this theorem has any implications for hidden variable models, we shall refrain from discussing that here ('t Hooft, 1999, 2000). It is just the mathematical fact that we wish to expose.

2. THE HARMONIC OSCILLATOR REVISITED

The treatment of the harmonic oscillator to be given here, differs somewhat from earlier treatments in this context (Blasone *et al.*, 2000; 't Hooft, 2000) although the philosophy is the same: we start with a deterministic system whose evolution law is represented by a quantum Hamiltonian.

The deterministic system we start with here is a set of N states, $\{(0), (1), \ldots, (N-1)\}$ on a circle. Time is discrete, the unit time steps having length τ (the continuum limit is left for later). The evolution law is

$$t \to t + \tau; \qquad (\nu) \to (\nu + 1 \bmod N). \tag{7}$$

On the basis spanned by the states (ν), the evolution operator is

$$U(\Delta t = \tau) = e^{-iHr} = e^{-\frac{\pi i}{N}} \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}. \tag{8}$$

The phase factor in front of the matrix is of little importance; it is there for future convenience. Its eigenstates are denoted as $|n\rangle$, $n = 0, \cdots, N-1$.

We have

$$H|n\rangle = \frac{2\pi\left(n + \frac{1}{2}\right)}{N_\tau}|n\rangle. \tag{9}$$

The $\frac{1}{2}$ comes from the aforementioned phase factor.

It is now instructive to apply the algebra of the $SU(2)$ generators L_x, L_y and L_z, so we write

$$N \stackrel{\text{def}}{=} 2\ell + 1, \qquad n \stackrel{\text{def}}{=} m + \ell. \qquad m = -\ell, \cdots, \ell.$$

Using the quantum numbers m rather than n to denote the eigenstates, we have

$$H|m\rangle = \frac{2\pi\left(m + \ell + \frac{1}{2}\right)}{(2\ell+1)\tau}|m\rangle \quad \text{or} \quad H = \frac{2\pi}{(2\ell+1)\tau}\left(L_z + \ell + \frac{1}{2}\right). \tag{10}$$

This Hamiltonian resembles the harmonic oscillator Hamiltonian when $\omega = 2\pi/(2\ell+1)\tau$, except for the fact that there is an upper bound for the energy. This upper bound disappears in the continuum limit, if $\ell \to \infty$, $\tau \downarrow 0$. Using L_x and L_y, we can make the correspondence more explicit.

Write

$$\begin{aligned} L_\pm|m\rangle &\stackrel{\text{def}}{=} \sqrt{\ell(\ell+1) - m(m\pm 1)}|m\pm 1\rangle; \\ L_\pm &\stackrel{\text{def}}{=} L_x \pm iL_y; \qquad [L_i, L_j] = i\varepsilon_{ijk}L_k, \end{aligned} \tag{11}$$

and define

$$x \stackrel{\text{def}}{=} \alpha L_x, \quad p \stackrel{\text{def}}{=} \beta L_y; \qquad \alpha \stackrel{\text{def}}{=} \sqrt{\frac{\tau}{\pi}}, \quad \beta \stackrel{\text{def}}{=} \frac{-2}{2\ell+1}\sqrt{\frac{\pi}{\tau}}. \tag{12}$$

The commutation rules are

$$[x, p] = \alpha\beta i L_z = i\left(1 - \frac{\tau}{\pi}H\right), \tag{13}$$

and since

$$L_x^2 + L_y^2 + L_z^2 = \ell(\ell+1), \tag{14}$$

we have

$$H = \frac{1}{2}\omega^2 x^2 + \frac{1}{2}p^2 + \frac{\tau}{2\pi}\left(\frac{\omega^2}{4} + H^2\right). \tag{15}$$

Now consider the continuum limit, $\tau \downarrow 0$, with $\omega = 2\pi/(2\ell+1)\tau$ fixed, for those states for which the energy stays limited. We see that the commutation rule (13) for x and p becomes the conventional one, and the Hamiltonian becomes that of the conventional harmonic oscillator. There are no other states than the legal ones, and their energies are bounded, as can be seen not only from (15) but rather from the original definition (10). Note that, in the continuum limit, both x and p become continuous operators.

The way in which these operators act on the "primordial" or "ontological" states (ν) of Eq. (7) can be derived from (11) and (12), if we realize that the states $|m\rangle$ are just the discrete Fourier transforms of the states (ν). This way, also the relation between the eigenstates of x and p and the states $((\nu)$ can be determined, but we will not dwell on these details.

The most important conclusion from this section is that there is a close relationship between the quantum harmonic oscillator and the classical particle moving along a circle. The period of the oscillator is equal to the period of the trajectory along the circle. We started our considerations by having time discrete, and only a finite number of states. This is because the continuum limit is a rather delicate one. One cannot directly start with the continuum

because then the Hamiltonian does not seem to be bounded from below.

3. MULTIPLE HARMONIC OSCILLATORS: THE FREE BOSE FIELD

Extending our procedure to a collection of many harmonic oscillators appears to be easy. We just take an equal number of particles moving on circles. To be accurate, we must take the time quantum τ equal for all circles. We then have an automaton, hopping from one state to the next at the beat of a clock. but how do we handle coupled oscillators?

If the oscillators are coupled harmonically, the prescription is easy: we diagonalize the Hamiltonian, and handle all normal modes independently. Then, we take as many circles as there are normal modes. But then the question remains: how do we recognize these circles in a realistic setting? In this paper, we set as our aim the understanding of the free Bose field, which is nothing but an infinite collection of harmonic oscillators, in terms of a deterministic model. The biggest challenge then is, how to arrive at a model that has a unique circular orbit for every normal mode of the quantum model.

Consider the Lagrangian (1), and the associated Klein–Gordon equation (2). The independent normal modes are the Fourier coefficients $\hat{\varphi}(\vec{k}, t)$, with

$$\varphi(\vec{x}, t) \stackrel{\text{def}}{=} (2\pi)^{-3/2} \int d\vec{k}\hat{\varphi}(\vec{k}, t)\, e^{i\vec{k}\cdot\vec{x}}. \tag{16}$$

If $\varphi(\vec{x}, t)$ is a classical field, then its Fourier modes $\hat{\varphi}(\vec{k}, t)$ are all classical oscillators. They are not confined to the circle, but the real parts, $\Re(\hat{\varphi}(\vec{k}, t))$, and the imaginary parts, $\Im(\hat{\varphi}(\vec{k}, t))$ of every Fourier mode each move in a two-dimensional phase space. If we want to reproduce the quantum system, we have to replace these two-dimensional phase spaces by one-dimensional circles.

It turns out to be possible to extract the circular component of these oscillations—and to remove their amplitudes!—but a certain amount of care is needed. We do not want to destroy translation invariance. This is why it is not advised to start from the real part, $\Re(\hat{\varphi}(\vec{k}))$, and the imaginary part, $\Im(\hat{\varphi}(\vec{k}))$, separately. Rather, we note that, at each $\vec{k}$, there are two oscillatory modes, a positive and a negative frequency. Thus, in general,

$$\begin{aligned} \hat{\varphi}(\vec{k}, t) &= A(\vec{k})e^{i\omega t} + B(\vec{k})e^{-i\omega t}; \\ \dot{\varphi}(\vec{k}, t) &= i\omega A(\vec{k})e^{i\omega t} - i\omega B(\vec{k})e^{-i\omega t}; \end{aligned} \tag{17}$$

where $\omega = (\vec{k}^2 + \mu^2)^{1/2}$. The most essential point of this paper now is that we have

to replace the amplitudes A and B by numbers of modulus one, keeping only the circular motions $e^{\pm i\omega t}$. A space translation would mix up the real and imaginary parts of $\hat{\varphi}(\hat{k}, t)$ and $\dot{\hat{\varphi}}(\vec{k}, t)$, which is why we use the decomposition (17), where a space translation merely rotates the phases $e^{\pm i\omega t}$.

We introduce the "gauge transformations"

$$\begin{aligned} A(\vec{k}) &\to R_1(\vec{k})A(\vec{k}), \\ B(\vec{k}) &\to R_2(\vec{k})B(\vec{k}), \end{aligned} \tag{18}$$

where $R_1(\vec{k})$ and $R_2(\vec{k})$ are real functions of $\vec{k}$. The *only* quantities invariant under these two transformations are the phases of A and B, which is what we want. In terms of φ and $\dot{\varphi}$, this transformation reads:

$$\begin{aligned} \varphi &\to \frac{R_1 + R_2}{2}\varphi + \frac{R_1 - R_2}{2i\omega}\dot{\varphi}, \\ \dot{\varphi} &\to \frac{R_1 + R_2}{2}\dot{\varphi} + \frac{i\omega(R_1 - R_2)}{2}\varphi. \end{aligned} \tag{19}$$

This is how we arrive at the transformation (5), with

$$\begin{aligned} \hat{K}_1(\vec{k}) &= \frac{R_1 + R_2}{2}, \\ \hat{K}_2(\vec{k}) &= \frac{R_2 - R_1}{2}\omega. \end{aligned} \tag{20}$$

4. DISCUSSION

If we define "observables" to be all functionals of the fields that are invariant under the transformations (3) with restrictions (4), then, as was derived in Section 3, at each Fourier mode, only the phase factors $e^{\pm i\omega t}$ are observable. These circular motions are the continuum limit of discrete circular motions, and the latter span the Hilbert space of states that are described by the $SU(2)$ algebra of operators L_x, L_y, and L_z. In the continuum limit, their Hamiltonian is that of the harmonic oscillator, and these harmonic oscillators combine into the system of quantized free Bose particles. In contrast to our previous constructions, there is no unwanted negative component of the Hamiltonian. It has been successfully projected out by our invariance requirement.

Extending our procedure to systems with multiple bosons, or vector bosons, may appear to be straightforward, except that the gauge group may become difficult to reconcile with continuous symmetries, such as rotation symmetry in a vector theory. Therefore, vector theories are still posing a challenge. Introducing interactions of any kind is an even bigger challenge.

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